# Pre-class Warm-up!!!

Which of the following results have we already seen in a previous section, to do with a path c : [a,b] -> R^n and a vector field F on R^n ?

a. If F = grad f then $\int_c F \cdot d\underline{s}$ = f(c(b)) - f(c(a)) Yes No

b. If $\int_c F \cdot d\underline{s}$ = f(c(b)) - f(c(a)) Yes No Not correct as written.

for some function f then F = grad f

c. If F = grad f then $\int_c F \cdot d\underline{s}$ does not depend on the particular choice of path from c(a) to c(b) Yes No

d. When n = 3, if F = grad f then curl F = 0 Yes No

e. $\int_c F \cdot d\underline{s}$ does not depend on the particular choice of path from c(a) to c(b) Yes No Incorrect statement.

## Section 8.3 Conservative vector fields

What we learn:

- a more complete angle on the gradient vector fields we have already studied
- path-independence
- another way to find a potential function whose gradient the field is
- a criterion for when a vector field is conservative
- slightly more elaborate applications, similar to what we have seen before

Theorem 7.
Let F be a vector field on R^n . The following are equivalent:

(ii) For any two oriented curves C_1 and C_2 that have the same end points $\int_{C_1} F \cdot d\underline{s} = \int_{C_2} F \cdot d\underline{s}$

(iii) F is the gradient of some function f

doesn't cross itself start = finish

(i) For any oriented simple closed curve C ,

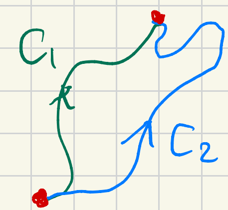
$$\int_C F \cdot d\underline{s} = 0$$

(iv) (assuming n = 3 ) curl F = 0

Definition: a vector field satisfying (ii) is called conservative

We have seen before?

- (ii) => (iii) Yes No
- (iii) => (ii) Yes No
- (i) => (ii) Yes No
- (ii) => (i) Yes No
- (iii) => (iv) Yes No
- (iv) => (iii) Yes No



Comments on:

(iii) $F$ is the gradient of some function $f$

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(iv) (assuming $n = 3$ ) curl $F = 0$

We don't really prove (iv) $\Rightarrow$ (iii)

If $F = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$ then

$$\nabla \times F = \left( \frac{\partial}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial y}, \quad , \quad \right)$$

Fact If these partial derivatives exist and are continuous then

$$\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y}$$

Thus $\nabla \times F = (0, 0, 0)$

2-dimensional version

If $F : \mathbb{R}^2 \to \mathbb{R}^2$ then

$F = (F_1, F_2) = \nabla f$

$\Leftrightarrow \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$

Recall: $\nabla \cdot (\nabla \times F) = 0$.

Theorem: When $n = 3$, given a vector field $F$ we can write

$F = \text{curl } G$ <=> Div $F = 0$

Questions (like 1 - 4, 17, 18):

Determine if the vector field
F(x,y,z) = (-2+4y, -4x, 0)
is a gradient vector field. If it is, find a function
f so that F = grad f.
Determine whether F = curl G for some
vector field G (but do not find G).

Solution

$$\nabla \times F = \left(\frac{\partial 0}{\partial y} - \frac{\partial(-4x)}{\partial z}, 0-0, -4-4\right)$$

$\neq 0$ so $F \neq \nabla f$ for some $f$.

$$\nabla \cdot F = \frac{\partial}{\partial x}(-2+4y) + \frac{\partial}{\partial y}(-4x) + \frac{\partial}{\partial z}(0)$$

$= 0$

so $F = \nabla \times G$ for some $G$.

Comments on:

(ii) For any two oriented curves C_1 and C_2 that have the same end points $\int_{C_1} F\cdot \underline{ds} = \int_{C_2} F\cdot \underline{ds}$

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(iii) F is the gradient of some function f

We have seen (iii) $\Rightarrow$ (ii). If $F = \nabla f$ then

$$\int_{\text{any } C} F = f(c(b)) - f(c(a))$$

is independent of choice of $C$.

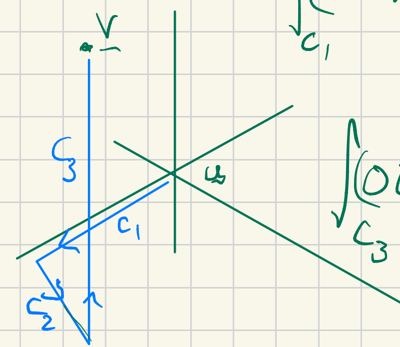
(ii) $\Rightarrow$ (iii) Assume (ii). We construct a function $f$ with $F = \nabla f$. Pick a point $\underline{u}$ in $\mathbb{R}^n$. For any vector $\underline{v}$ define $f(\underline{v}) = \int_C F \cdot \underline{ds}$ where $C$ is any path from $\underline{u}$ to $\underline{v}$.

We have to show $\nabla f = F$.

Consider $F_3 \overset{?}{=} \frac{\partial}{\partial z} f$

$$\int_{C_1} (0\ 0\ F_3)\, \underline{ds} = \int_{C_2} (0\ 0\ F_3) \cdot \underline{ds}$$

$$= 0.$$




$$\int_{C_3} (0\ 0\ F_3) \cdot \underline{ds} = \int_{C_3} F_3 \, dz$$

is an antiderivative of $F_3$, $\frac{\partial f}{\partial z} = F_3$

Do the same with components $F_1$ and $F_2$, and add.

We have a new way to compute a potential function f for F when F is conservative, but in practice it is not an improvement on the way we have already seen.

Example: Find f so that F = grad f when F(x,y,z) = (e^x sin y, e^x cos y, z^2).

Solution: Use method we already know:

Solve $\frac{\partial f}{\partial x} = e^x \sin y$ $\quad \frac{\partial f}{\partial y} = e^x \cos y$

$\frac{\partial f}{\partial z} = z^2$

$$f = e^x \sin y + \frac{1}{3} z^3 .$$

Example: Find $\int_c F \cdot d\underline{s}$

when c(t) = (t, e^(sin t)), 0 ≤ t ≤ π) and F(x,y,z) = (y, x)

Solution

$F = \nabla f$ where


$f(x,y) = xy$

$$\int_c F \cdot ds = \pi \cdot 1 - 0 \cdot 1 = \pi .$$

Comments on

(ii) For any two oriented curves C_1 and C_2 that have the same end points

$$\int_{C_1} F \cdot d\underline{s} = \int_{C_2} F \cdot d\underline{s}$$

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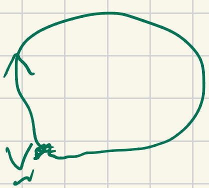
(i) For any oriented simple closed curve C ,

$$\int_C F \cdot ds = 0$$

Proof of (ii) $\Rightarrow$ (i). Suppose (ii)

Let $C$ be a curve from $\underline{v}$ to $\underline{v}$



Take $C_1 = C$

$C_2 =$ the path that is constant at v.

$$\int_{C_2} F \cdot d\underline{s} = 0 \quad b/c \quad c_2' = 0$$

$$= \int_C F \cdot d\underline{s}$$